*On possible Periodic Inequalities in the Epoch of the Sun-spot Variation. Papers of the I.U.S.R. Computing Bureau, No. III.*
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1. When any point of importance arises with regard to the stars, it is natural to inquire whether it is represented in the behaviour of the one star which we can observe to special advantage, viz. our own Sun. Reasons have been given elsewhere * for regarding the Sun, provisionally at any rate, as a long-period variable with a period of $11\frac{1}{8}$ years, but with a very small range of variation. Reasons have also been given † for regarding the oscillations of maximum of long-period variables as systematic; so that when Chandler writes for the epochs of maximum an expression such as

$$\epsilon + \mathrm{P.E} + \mathrm{C} \sin (\mathrm{A}^\circ.\mathrm{E} + \alpha),$$

where P is the period in days, and E the number of periods elapsed since the epoch $\epsilon$, and C, A, $\alpha$ three constants, we have approximately

$$\mathrm{C} = 7^{\mathrm{d}}\cdot3 + \cdot064\ \mathrm{P}$$
$$\mathrm{A} = 2^\circ\cdot6 + \cdot018\ \mathrm{P}.$$

Hence the questions arise,—Is there an oscillation of this kind in the Sun-spot maxima? and if so, how far does it fit in with these formulæ? The period of the Sun (say 4060 days) is so very much longer than that of any of the stars used in deriving the above formulæ that we may be prepared for some error, even a considerable error, in the extrapolation; but we may at least expect both A and C to be larger than for the stars, if the Sun falls into line at all.

2. Dr. Wolfer has given (on page 96 of the *Astron. Mitteilungen*, No. xciii.) a list of deviations of 26 minima and 26 maxima from a uniform arithmetical progression. These are shown in Table I.

TABLE I.

*Wolfer's Minima and Maxima of Sun-spots.*

| Min. | Max. | Min. | Max. | Min. | Max. |
|---|---|---|---|---|---|
| y | y | y | y | y | y |
| +0·3 | −0·8 | +1·2 | +2·1 | −0·5 | +0·5 |
| −2·7 | −1·4 | +1·6 | +0·3 | +1·1 | +2·9 |
| +1·2 | +1·0 | +0·9 | +0·4 | +0·6 | −0 9 |
| +1·0 | −0·6 | +0·8 | +0·9 | −1·0 | −1·1 |
| −0·1 | −0·6 | −0·2 | +1·0 | +0·4 | −0·2 |
| −0·2 | +3·3 | 0·0 | −1·9 | +0·4 | −0·8 |
| +2·1 | +2·2 | −2·1 | −4·2 | +1·0 | +1·4 |
| +1·0 | −0·9 | −4·1 | −5·6 | +0·5 | +0·5 |
| −1·7 | +0·5 | −1·6 | +0·4 | | |

(The unit is one year.)

* See *Mon. Not.*, lxiv. p. 549 and lxvii. p. 334.
† *Mon. Not.*, lxviii. p. 544 (June 1908).

3. The problem is to find whether these residuals are affected with a periodicity C sin $(A^\circ.E + \alpha)$, of which both the coefficient C and the argument A are unknown. In such a case, Professor Schuster has insisted that we have no resource but to try *all* values of A; or rather a series of values separated by small intervals, the size of the interval depending on the "resolving power" of the available material. In the present case, successive values of A differing by 2° were tried, and the resulting "periodogram" shows that this interval is sufficiently small. For each value of A we form

$$a^2 + b^2 \equiv \left(\sum r_E \sin A^\circ.E\right)^2 + \left(\sum r_E \cos A^\circ.E\right)^2$$

E being used by Chandler to denote the number of periods after a given epoch, and $r_E$ being the corresponding residual in Table I. If we had *exactly*

$$r_E = C \sin (A^\circ.E + \alpha)$$

then we should have

$$a = C\left[\cos \alpha \sum \sin^2 AE + \sin \alpha \sum \sin AE \cos AE\right]$$

$$= \tfrac{1}{2}C\left[\cos \alpha \sum (1 - \cos 2AE) + \sin \alpha \sum \sin 2AE\right]$$

$$= \tfrac{1}{2}C\left[n \cos \alpha - \sum \cos (2AE + \alpha)\right]$$

where $n$ is the number of terms in the series—viz. 26 in our case. Now the sum $\Sigma \cos (2AE + \alpha)$ is generally small compared with $n$. It will be seen later that the smallest value of A which seriously concerns us is about 20°; so that as E increases from 0 to 26, 2AE increases from 0° to $40^\circ \times 26 = 3 \times 360^\circ$ nearly. Hence $\cos (2AE + \alpha)$ runs through its cycle nearly three times; and since in each cycle the positive and negative terms cut one another out, what is left is small. Neglecting it, we should have, in the ideal case,

$$a = \tfrac{1}{2}C\, n \cos \alpha \quad \text{and} \quad b = \tfrac{1}{2}C\, n \sin \alpha$$

$$\text{or} \qquad 13C = \sqrt{a^2 + b^2}.$$

But accidental errors will give a sensible value for $a^2 + b^2$ for any value of A. We must look for values of A which give *large* values of $a^2 + b^2$. These values are tabulated in Table II., which was calculated in the Computing Bureau of the I.U.S.R.

TABLE II.

*Periodogram of Wolfer's Residuals.*

| A | Min. | Max. | A | Min. | Max. | A | Min. | Max. |
|---|---|---|---|---|---|---|---|---|
| 2° | ·00 | ·04 | 48 | 1·06 | 1·97 | 94 | 1·04 | 1·08 |
| 4 | ·01 | ·09 | 50 | 1·01 | 2·81 | 96 | ·92 | ·92 |
| 6 | ·07 | ·16 | 52 | 1·05 | 3·52 | 98 | ·75 | ·80 |
| 8 | ·26 | ·39 | 54 | ·95 | 3·34 | 100 | ·82 | ·53 |
| 10 | ·33 | ·70 | 56 | ·66 | 2·98 | 102 | ·72 | ·28 |
| 12 | ·45 | ·99 | 58 | ·43 | 2·57 | 104 | ·66 | ·46 |
| 14 | ·69 | 1·51 | 60 | ·41 | 2·39 | 106 | ·60 | ·52 |
| 16 | 1·27 | 1·80 | 62 | ·26 | 1·70 | 108 | ·57 | ·59 |
| 18 | 1·83 | 2·16 | 64 | ·27 | 1·30 | 110 | ·44 | ·77 |
| 20 | 1·82 | 2·56 | 66 | ·36 | 1·13 | 112 | ·40 | ·83 |
| 22 | 1·76 | 2·50 | 68 | ·46 | ·95 | 114 | ·35 | ·72 |
| 24 | 1·60 | 2·26 | 70 | ·45 | ·95 | 116 | ·26 | ·62 |
| 26 | 1·52 | 2·13 | 72 | ·47 | ·78 | 118 | ·27 | ·40 |
| 28 | 1·07 | 1·59 | 74 | ·28 | ·80 | 120 | ·22 | ·28 |
| 30 | 0·78 | 1·04 | 76 | ·13 | ·48 | 122 | ·17 | ·21 |
| 32 | 1·52 | ·92 | 78 | ·19 | ·75 | 124 | ·19 | ·29 |
| 34 | ·50 | ·56 | 80 | ·29 | ·97 | 126 | ·31 | ·34 |
| 36 | ·46 | ·43 | 82 | ·48 | 1·23 | 128 | ·20 | ·48 |
| 38 | ·50 | ·41 | 84 | ·71 | ·93 | 130 | ·16 | ·48 |
| 40 | ·56 | ·62 | 86 | ·85 | 1·09 | 132 | ·14 | ·44 |
| 42 | ·77 | ·91 | 88 | ·98 | 1·06 | 134 | ·10 | ·28 |
| 44 | ·94 | 1·25 | 90 | ·87 | 1·15 | 136 | ·10 | ·16 |
| 46 | ·96 | 2·12 | 92 | 1·08 | 1·12 | 138 | ·23 | ·09 |
| Means | 0·86 | 1·18 | | 0·60 | 1·56 | | 0·41 | 0·50 |

4. We have to look for cases where the value of $a^2+b^2$ strikingly exceeds its mean value. The mean value should, however, be taken from the undisturbed portions of the periodogram, and there is always a little doubt in the first instance which these are. The means of the columns have been taken, and the mean of the six is 0·82. The highest value reached, 3·52, is only 4·4 times this mean; but if we exclude the column in which it occurs the mean falls to ·71, and the highest value, 3·52, is very nearly five times this. There is just a possibility that this peak indicates a real periodicity, though the probability is not high. It will be seen that there is a slight rise in the column for minima near the same value of A (viz. 52°), though the rise is far too slight to be worth notice independently. Still, there are obvious reasons why the minimum may be not so well determined as the maximum.

5. There is also a rise at A = 20°, though not quite so marked.

It is worth noticing, however, that the accompanying rise in the column for minima is greater, and that there are indications in some of the variable stars of another term accompanying the main inequality (see the case of S Serpentis in *M.N.*, lxviii. p. 563).

6. To state the case as favourably as possible for the existence of a periodic inequality in the sun-spot period, we may exclude from both columns the values

$$\text{from } A = 14^\circ \text{ to } A = 30^\circ, \text{ and from } A = 44^\circ \text{ to } A = 66^\circ,$$

and then the mean value for the periodogram for minima becomes 0·46 and for the maximum 0·62. The ratio of 3·52 to its own mean is thus nearly 6, and to the mean for both maximum and minimum is nearly 7.

7. Recurring now to the quantities which give us the inequality, if it exists, we find for the formulæ representing the maxima, when $A = 53^\circ$ (which seems the best value),

$$-1{\cdot}4 \sin 53^\circ E - 0{\cdot}2 \cos 53^\circ E,$$

and for the minima

$$-0{\cdot}7 \sin 53^\circ E - 0{\cdot}1 \cos 53^\circ E.$$

But the epochs are not quite the same. The minima begin with 1610·5, and the maxima with 1616·3, or 5·8 years later, which is $5{\cdot}8/11{\cdot}13 = {\cdot}52$ of a period. The phase of the inequality is thus advanced by $53^\circ \times {\cdot}52 = 27^\circ$ for the maxima, and to compare the expressions we must write

$$-1{\cdot}4 \sin (53^\circ E - 27^\circ) - 0{\cdot}2 \cos (53^\circ E - 27^\circ)$$

$$= -1{\cdot}3 \sin 53^\circ E + 0{\cdot}4 \cos 53^\circ E \quad \text{for maxima,}$$

and, as before,

$$-0{\cdot}7 \sin 53^\circ E - 0{\cdot}1 \cos 53^\circ E \quad \text{for the minima.}$$

8. In Chandler's notation, the sun-spot maxima would therefore be given by some such formula as

$$\text{const} + 4060\ E + 490 \sin (53^\circ E + \text{const}),$$

and the question now arises how far this formula for the Sun accords with those found for the stars and quoted in § 1. Putting $P = 4060^d$ in these formulæ, we have

$$C = 7^d{\cdot}3 + 0^d{\cdot}064 \times 4060 = 267^d \text{ as against } 490$$

$$A = 2^\circ{\cdot}6 + 0^\circ{\cdot}018 \times 4060 = 76^\circ \text{ as against } 53^\circ.$$

There is, of course, no reason why these formulæ should be strictly linear, so far as we know at present; and if we remark that the value of the coefficient C obtained from the minima is only about half that obtained from the maxima (*i.e.* in sensible agreement with the star-formula), the accordance is sufficiently good

to suggest further inquiry. One fact emerges from the discussion, viz. that from the available material it is difficult to make sure of the existence of an inequality similar to those shown by the stars. We may take it that the coefficient C is of the right order of magnitude to fit in with the star-formula: and our periodogram shows that in this case it is too small to stand out clearly from the accidental inequalities. It will not be possible to affirm or deny the existence of such an inequality with confidence until the material is improved by extending the series of observations, or possibly by reducing the accidental errors of the older observations by improved discussions of them. One hope of reducing the accidental errors proved vain. It was thought that, since there are two independent series of maxima and minima, they might be used in combination in some way, so that the effect of accidental errors of one series might be reduced by the other. But apparently the two series run together so closely that not much can be gained in this way.

9. The value $A=20°$ gives a value of C not much smaller than that for $A=53°$; and from the material it is not easy to say which of these two possible terms corresponds to the terms found for variable stars. Are there possibly two terms in general? The case of S Serpentis has already been quoted, where the existence of a long-period term had masked the short-period term. In other cases there may be long-period terms affecting the short-period terms to a smaller extent, and this may account for some of the large deviations from the formula. And these two values $A=53°$ and $A=20°$ for the Sun may help us, by suggestion, in getting at the facts for the variables. The series of observed maxima for the Sun is much longer and more continuous than those for most of the variables; and it would not be surprising if we got suggestions from it which would help in elucidating the shorter series.

10. Assuming that the value $A=53°$ corresponds to the terms that have attracted attention for the variables, then the formula

$$A = 2°{\cdot}6 + 0°{\cdot}018 \times P$$

does not hold for so large a value of P as $P=4060$; for which we should get $A=75°$, as remarked in § 7. Can this distant point on the curve be used to improve the formula?

11. Firstly, let us examine the consequence of assuming the formula still linear, and let us determine $a$ and $b$ in the expression

$$A = a + bP$$

so as to satisfy the Sun and the mean of the stars; that is, put

$$53° = a + b \times 4060 \qquad \text{(the Sun)}$$

$$8°{\cdot}16 = a + b \times 311 \qquad \text{(mean of stars).}$$

From these we get $b = 0°{\cdot}012,\ a = 4°{\cdot}43.$

This new formula $A_2 = 4^\circ{\cdot}4 + 0^\circ{\cdot}012$ P
differs from the former $A_1 = 2{\cdot}6 + 0{\cdot}018$ P

as follows, for different values of P :—

| | P = | 100^d | 200^d | 300^d | 400^d | 500^d | 600^d |
|---|---|---|---|---|---|---|---|
| New formula | $A_2 =$ | 5°·6 | 6°·8 | 8°·0 | 9°·2 | 10°·4 | 11°·6 |
| Old formula | $A_1 =$ | 4·4 | 6·2 | 8·0 | 9·8 | 11·6 | 13·4 |
| | $A_2 - A_1 =$ | +1·2 | +0·6 | 0·0 | −0·6 | −1·2 | −1·8 |

It seems doubtful whether our present material is sufficient to enable us to discriminate between these two formulæ, for most of the stars have periods between 200 and 400 days. But we may notice one other supposition, viz.—

12. Secondly, let us adopt the suggestion of a curve of some kind rather than a straight line. The appropriate indices for A and P will be suggested by finding $m$ in the formula

$$A^m = P$$

for the large values of A and P, *i.e.* for the case of the Sun. We have

$$m = \log P/\log A = \log 4060/\log 53$$
$$= 3{\cdot}61/1{\cdot}72 = 2{\cdot}1.$$

This suggests some formula such as either

$$A^2 = a(P + p)$$

$$\text{or} \qquad (A + a)^2 = bP.$$

13. Determining the constants from the two cases of the Sun and mean of the stars, we find for the two suppositions

$$A^2 = 0{\cdot}73\ (P - 220)$$
$$(A + 9^\circ)^2 = 0{\cdot}94\,P.$$

Of these, the former gives impossible values of A for periods below 220 days, and is thus unsuitable. The latter gives values of A for different values of P, as below.

| P = | 100^d | 200^d | 300^d | 400^d | 500^d | 600^d | |
|---|---|---|---|---|---|---|---|
| $A_3 =$ | 0°·7 | 4°·7 | 7°·8 | 10°·4 | 12°·7 | 14°·8 | (new formula). |
| $A_1 =$ | 5·6 | 6·8 | 8·0 | 9·2 | 10·4 | 11·6 | (linear formula). |
| $A_3 - A_1 =$ | −4·9 | −2·1 | −0·2 | +1·2 | +2·3 | +3·2. | |

These differences are larger than the former, and it seems probable that we can discriminate even now in favour of the original formula.

SUMMARY.

The sun-spot maxima (and minima) occur on the average at intervals of 11·125 years. But the individual maxima (and minima) show discordances which have been tabulated for 26 periods by Wolfer. Analysing these by the periodogram method of Professor Schuster, there are indications of two periodicities, one of which the phase advances 53° per period of $11\frac{1}{8}$ years, the other of which the phase advances only 20°. The cycles are completed in about 75 and 200 years respectively. The amplitude of each inequality is about a year, but the accidental errors are so large that either or both of these inequalities may be spurious.

The quicker moving inequality (53°) can be brought into line with similar inequalities for the long-period variables; the best formula connecting A (advance of phase in degrees per period) with P (the period in days) being the simple linear formula

$$A = 4^{\circ}\cdot 4 + 0^{\circ}\cdot 012\ P.$$

The slower moving inequality (20°) may quite possibly have analogies in the stars, but as yet the material is not sufficient to declare.


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## *Researches on the Solar Constant and the Temperature of the Sun.* By Dr. J. Scheiner, Assoc. R.A.S.

In No. 55 of the Publications of the Astrophysical Observatory, Potsdam, I have published an extended paper on this subject, and I should like to give a short report of the results to the readers of the *Monthly Notices.*

The measures of the Sun's radiation were made with the Ångström Electric Compensation Pyrheliometer, to which I had given a modified exterior form and a parallactic motion with clockwork. On eleven days in June and July 1903 I made a long series of observations on the top of the Gorner Grat in Canton Wallis (Switzerland), from which I could derive the radiation of the Sun outside our atmosphere. This part of the problem is the most difficult one, and, according to my view, it cannot be solved from measurements of the solar radiation alone. From such observations a portion only of the real solar constant can be obtained, because only that portion of the loss by absorption in our atmosphere can be calculated which is based upon the *continuous* increase of absorption with growing thickness of the atmospheric layer traversed by the radiation. With carbon dioxide and water vapour there exists a nearly sudden absorption in the highest thin layers of the atmosphere, which must be treated as a constant to be added to the